Name: _____

- This packet is due at **12:00 pm PDT (noon) on Friday, April 30**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- Make sure that it is clear what your final answer is. This may mean boxing/circling it or it may mean doing neither.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- I will not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Use the limit comparison test to determine whether each of the following series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

$a_n = \frac{n^2}{n^3+1} > 0, \quad b_n = \frac{1}{n} > 0$

$\lim_{n\to\infty} \frac{\left(\frac{n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = \lim_{n\to\infty} \frac{3n^2}{3n^2} = 1$ ($\frac{\infty}{\infty}$)

By the L.C.T., $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ has the same behavior as $\sum_{n=1}^{\infty} \frac{1}{n}$. But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞ by the p-test ($p=1$), so $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ also diverges to ∞

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3-2}}$

$a_n = \frac{1}{\sqrt{n^3-2}} > 0$ for $n \geq 2$, $b_n = \frac{1}{n^{3/2}} > 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n^3-2}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n\to\infty} \frac{n}{(n^3-2)^{1/2}}$

$= \lim_{n\to\infty} \left(\frac{n^3}{n^3-2}\right)^{1/2} = \left(\lim_{n\to\infty} \frac{n^3}{n^3-2}\right)^{1/2}$ ($\frac{\infty}{\infty}$)

$= \left(\lim_{n\to\infty} \frac{3n^2}{3n^2}\right)^{1/2} = 1^{1/2} = 1$

By the L.C.T., $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3-2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ have the same behavior. But $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-test ($p = 3/2$), so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3-2}}$ also converges.

(c) $\sum_{n=1}^{\infty}\left(\frac{\ln(n)}{n}\right)^2$

$a_n = \left(\frac{\ln(n)}{n}\right)^2 > 0, \quad b_n = \frac{1}{n^{3/2}} > 0$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(\frac{\ln(n)}{n}\right)^2}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n\to\infty}\frac{n^{3/2}\ln(n)^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{\ln(n)^2}{n^{1/2}} = \lim_{n\to\infty}\frac{2\ln(n)\cdot\frac{1}{n}}{\frac{3}{2}n^{1/2}}$$

$\frac{\infty}{\infty}$

$$= \lim_{n\to\infty}\frac{2\ln(n)}{\frac{3}{2}n^{3/2}} = \lim_{n\to\infty}\frac{2\cdot\frac{1}{n}}{\frac{3}{4}\cdot n^{1/2}} = \lim_{n\to\infty}\frac{2}{\frac{3}{4}\cdot n^{3/2}}$$

$$= 0$$

Since $\sum_{n=1}^{\infty} b_n$ converges by the p-test $(p=\frac{3}{2})$,

$\sum_{n=1}^{\infty} a_n$ also converges.

(d) $\sum_{n=1}^{\infty}\frac{1}{n^2 - n\sin(1/n)}$

$a_n = \frac{1}{n^2 - n\sin(\frac{1}{n})} > 0, \quad b_n = \frac{1}{n^2} > 0$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(\frac{1}{n^2-n\sin(\frac{1}{n})}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty}\frac{n^2}{n^2-n\sin(\frac{1}{n})}$$

$$= \lim_{n\to\infty}\frac{n^2}{n^2\left(1-\frac{1}{n}\cdot\sin(\frac{1}{n})\right)} = \lim_{n\to\infty}\frac{1}{1-\left(\frac{1}{n}\right)\sin(\frac{1}{n})} = 1$$

$\left(\frac{1}{n}\right)\sin(\frac{1}{n}) \to 0$ by squeeze theorem

By the L.C.T. $\sum_{n=1}^{\infty}\frac{1}{n^2-n\sin(\frac{1}{n})}$ has the same behavior as $\sum_{n=1}^{\infty}\frac{1}{n^2}$. But $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-test $(p=2)$, so $\sum_{n=1}^{\infty}\frac{1}{n^2-n\sin(\frac{1}{n})}$ also converges

(e) $\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$

$a_n = \frac{1}{4^n - 3^n} > 0, \quad b_n = \frac{1}{4^n} > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{4^n - 3^n}\right)}{\left(\frac{1}{4^n}\right)} = \lim_{n\to\infty} \frac{4^n}{4^n - 3^n}$$

$$= \lim_{n\to\infty} \frac{4^n}{4^n\left(1 - \left(\frac{3}{4}\right)^n\right)} = \lim_{n\to\infty} \frac{1}{1 - \left(\frac{3}{4}\right)^n} = 1$$

$\left(\frac{3}{4}\right)^n \to 0$

By the L.C.T. $\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$ has the same behavior as $\sum_{n=1}^{\infty} \frac{1}{4^n}$ which converges since it is a geometric series with $r = \frac{1}{4}$. So $\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$ also converges.

(f) $\sum_{n=1}^{\infty} \frac{6^n - 1}{5^n}$

$a_n = \frac{6^n - 1}{5^n} > 0, \quad b_n = \left(\frac{6}{5}\right)^n > 0$

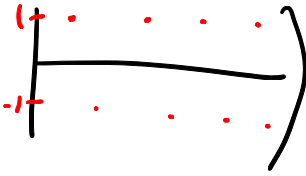
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{6^n - 1}{5^n}}{\left(\frac{6}{5}\right)^n} = \lim_{n\to\infty} \frac{\left(\frac{6}{5}\right)^n \cdot \left(1 - \frac{1}{6^n}\right)}{\left(\frac{6}{5}\right)^n}$$

$$= \lim_{n\to\infty} 1 - \frac{1}{6^n} = 1$$

By the L.C.T., $\sum_{n=1}^{\infty} \frac{6^n - 1}{5^n}$ has the same behavior as $\sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$, which is a geometric series with $r = \frac{6}{5}$, so it diverges. Hence, $\sum_{n=1}^{\infty} \frac{6^n - 1}{5^n}$ also diverges.

2. State whether the following series converges absolutely, conditionally, or not at all.

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+3}$

$\lim_{n\to\infty} \frac{n}{n+3} = 1$, so $\lim_{n\to\infty} (-1)^{n+1}\cdot\frac{n}{n+3}$ does not exist (graph looks something like )

Hence, $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n}{n+3}$ diverges by the divergence test

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$

$0 \le \left|\frac{(-1)^{n+1}}{n!}\right| = \frac{1}{n!} \le \frac{1}{2^n}$ for $n \ge 2$

But $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges since it is geometric with $r = \frac{1}{2}$

So $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges by the D.C.T.

Hence, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$ converges absolutely

(c) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{1+n^4}$

$0 \le \left|(-1)^{n+1}\frac{n^2}{1+n^4}\right| = \frac{n^2}{1+n^4} \le \frac{n^2}{n^4} = \frac{1}{n^2}$

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-test ($p=2$), so

$\sum_{n=1}^{\infty}\frac{n^2}{1+n^4}$ also converges. Hence $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{n^2}{1+n^4}$

converges absolutely.

(d) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges to ∞ by the p-test ($p=1$)

But $\frac{(-1)^n}{n}$ is alternating with $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$

converging to 0 monotonically (since $\frac{1}{n+1} \le \frac{1}{n}$)

So by the A.S.T. $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$ converges

conditionally.

(e) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n+3}}{n}$

$\left|(-1)^{n+1}\frac{\sqrt{n+3}}{n}\right| = \frac{\sqrt{n+3}}{n} \to \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$

Since $\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges, so does $\sum_{n=1}^{\infty}\frac{\sqrt{n+3}}{n}$ (by the D.C.T.)

Note that $(-1)^{n+1}\frac{\sqrt{n+3}}{n}$ is alternating and $\frac{\sqrt{n+3}}{n} = \sqrt{\frac{n+3}{n^2}} = \sqrt{\frac{1}{n}+\frac{3}{n^2}}$ is decreasing monotonically to 0, so by the A.S.T. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n+3}}{n}$

Converges conditionally.

(f) $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$

$\lim_{n\to\infty} 2^{1/n} = 1$, so $\lim_{n\to\infty}(-1)^{n+1}2^{1/n}$

does not exist.

By the divergence test, $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$

diverges.

3. In each of the following problems, use the estimate $|R_k| \leqslant b_{k+1}$ to find a value of k that guarantees that the sum of the first k terms of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_k for this k.

(a) $b_n = 1/\sqrt{n}$, error $< 10^{-3}$

Want: $\frac{1}{\sqrt{k+1}} < 10^{-3} \iff 10^3 < \sqrt{k+1}$

$\iff 10^6 < k+1$

$\iff 10^6 - 1 < k$

Take $k = 10^6 = 1000000$, so

$|R_{1000000}| \leq |b_{1000000}| < 10^{-3}$

$$S_{1000000} = \sum_{n=1}^{1000000} \frac{(-1)^{n+1}}{\sqrt{n}} \approx 0.60$$

(b) $b_n = 1/2^n$, error $< 10^{-6}$

Want $\frac{1}{2^{k+1}} < 10^{-6} \iff 10^6 < 2^{k+1}$

$\iff \log_2(10^6) < k+1$

$\iff \log_2(10^6) - 1 < k$

19.93

Take $k = 19$

$$S_{19} = \sum_{n=1}^{19} \frac{(-1)^{n+1}}{2^n} \approx 0.333334$$

(c) $b_n = \ln\left(1+\frac{1}{n}\right)$, error $< 10^{-3}$

Want: $\ln\left(1+\frac{1}{k+1}\right) < 10^{-3} \iff 1+\frac{1}{k+1} < e^{10^{-3}}$

$\iff \frac{1}{k+1} < e^{10^{-3}} - 1$

$\iff k+1 > \frac{1}{e^{10^{-3}}-1}$

$\iff k > \frac{1}{e^{10^{-3}}-1} - 1$

≈ 998.5

Take $k = 999$

$$S_{999} = \sum_{n=1}^{999} (-1)^{n+1} \ln\left(1+\frac{1}{n}\right) \approx 0.452$$

(d) $b_n = 1/n^2$, error $< 10^{-6}$

Want $\frac{1}{(k+1)^2} < 10^{-6} \iff 10^6 < (k+1)^2$

$\iff 10^3 < k+1$

$\iff 10^3 - 1 < k$

Take $k = 10^3 = 1000$, so $|R_{1000}| \le b_{1000} < 10^{-3}$

$$S_{1000} = \sum_{n=1}^{1000} \frac{(-1)^{n+1}}{n^2} \approx 0.822467$$

4. Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

Note that $\sum_{n=1}^{\infty} |a_n b_n| = \sum_{n=1}^{\infty} a_n$, which converges. Since $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely it converges (by the Absolute Convergence Test)